

Mapping Question Items to Skills with Non-negative Matrix Factorization

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ABSTRACT

Intelligent learning environments need to assess the student skills to tailor course material, provide helpful hints, and in general provide some kind of personalized interaction. To perform this assessment, question items, exercises, and tasks are presented to the student. This assessment relies on a mapping of tasks to skills. However, the process of deciding which skills are involved in a given task is tedious and challenging. Means to automate it are highly desirable, even if only partial automation that provides supportive tools can be achieved. A recent technique based on Non-negative Matrix Factorization (NMF) was shown to offer valuable results, especially due to the fact that the resulting factorization allows a straightforward interpretation in terms of a Q-matrix. We investigate the factors and assumptions under which NMF can effectively derive the underlying high level skills behind assessment results. We demonstrate the use of different techniques to analyze and interpret the output of NMF. We propose a simple model to generate simulated data and to provide lower and upper bounds for quantifying skill effect. Using the simulated data, we show that, under the assumption of independent skills, the NMF technique is highly effective in deriving the Q-matrix. However, the NMF performance degrades under different ratios of variance between subject performance, item difficulty, and skill mastery. The results corroborates conclusions from previous work in that high level skills, corresponding to general topics like World History and Biology, seem to have no substantial effect on test performance, whereas other topics like Mathematics and French do. The analysis and visualization techniques of the NMF output, along with the simulation approach presented in this paper, should be useful for future investigations using NMF for Q-matrix induction from data¹.

1. INTRODUCTION

Skills determine the outcome of an individual over a given task, namely whether the individual's task result will be a success or a failure. A task can be a question item, an exercise, or any challenge that will require some skills. This mapping of skills to tasks, or items, is a fundamental requirements for intelligent learning environments, since the

¹This article is a revised version of [7] published in the Proceedings of the Educational Data Mining Conference, 2011.

goal of these environments is to bring the learner to a target level of mastery over a set of skills. This goal relies on such mapping to assess the student's mastery level, and to decide which are the most appropriate exercises given this assessment. In the field of Educational data mining [2], this mapping is referred to as a Q-matrix.

Determining which skills are involved in a task can prove both tedious and difficult. The construction of a Q-matrix from data is a highly desirable goal for tutoring systems. Not only would it waive the expertise and labour intensive task of assigning which skills are involved in which task, but it would also offer a more objective and replicable means of getting the correct skill-to-task mapping. Furthermore, it might also allow a more effective means to build Q-matrices, as machine learning methods often outperform humans over a range of complex tasks.

However, the success in achieving this goal remains limited. Nowadays, we find no reliable method to automate the mapping of skills to tasks from data, but some progress has been made.

Working with log data from tutoring systems, data which is characterized by the fact that the knowledge state of the student dynamically changes in the data as the student learns, Cen et al. [6, 5] have used a technique known as Learning Factor Analysis (LFA) in order to bring improvements over an initially hand built Q-matrix (also termed a *transfer model*). This technique was shown useful for bringing improvements to the Q-matrix composed of fine-grained skills which are deemed necessary to complete certain exercises.

Barnes [3] developed a method of mapping skills to items based on a measure of the fit of a potential Q-matrix to the data. This method is inspired from the work of Tatsuoka [12]. It relies on static student knowledge states, as opposed to the dynamically changing knowledge states of the LFA technique. The methods described in this article also rely on static knowledge states. Barnes' method is fully automated and it was shown to perform at least as well as Principal Component Analysis for skill clustering analysis. However, it involves an algorithm that does not scale well to a Q-matrix that comprises 20 or more items.

Winters et al. [13] investigated how a number of standard clustering techniques can effectively match skills to test items. They applied these techniques to a wide array of test outcomes, from SAT topics such as Mathematics, Biology and French, to computer science exams, and to different trivia topics. Their findings show that for skills associated to topics within a single course, for example, the techniques were essentially no better at classifying test items than random

clustering. The same conclusion applies for topics like World history and Biology. However, the techniques were relatively successful at separating items that belongs to totally different topics, such as Mathematics and French.

In this paper, we replicate parts of the study by Winters et al. [13] and focus on one of the cluster algorithms they used, Non-negative Matrix Factorization (NMF). We use visualization techniques to analyze in greater details the results of the factorization. We propose a model to simulate student data and show that the NMF technique is indeed effective under certain assumptions. We use the simulation data model parameters as a means to quantify and estimate the effect of skills over the observed examinee performance in some of the real data of Winters et al. original study. First, let us give some details about NMF.

2. NON-NEGATIVE MATRIX FACTORIZATION AND Q-MATRIX INTERPRETATION

Non-negative matrix factorization (NMF) decomposes a matrix into two smaller matrices. It is used for dimensionality reduction, akin to Principal Component Analysis and Factor analysis. NMF decomposes a matrix of $n \times m$ positive numbers, \mathbf{V} , as the product of two matrices:

$$\mathbf{V} \approx \mathbf{WH} \tag{1}$$

The matrices \mathbf{W} and \mathbf{H} are respectively $n \times r$ and $r \times m$, where r is called the rank of the factorization. For our purpose, matrix \mathbf{V} represents the observed test outcome data for n question items and m respondents. Therefore, the product of \mathbf{W} and \mathbf{H} reproduces the observed patterns of success/failures of the m examinee to the n items. The matrix \mathbf{W} can be considered as a Q-matrix, whereas \mathbf{H} can be considered as the skills mastery for each m examinee. In the case of a Q-matrix, r represents the number of skills, which can take any value but should normally conform to: $r < nm/(n + m)$ [9].

Let us take an example to better explain NMF in our context. Assume the following Q-matrix, \mathbf{W} , composed of 3 skills and 4 items, and the following skills mastery matrix, \mathbf{H} , for 5 examinees:

$$\mathbf{W} = \begin{matrix} & \text{skills} \\ \text{items} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \quad \mathbf{H} = \begin{matrix} & \text{examinees} \\ \text{skills} & \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$

Given this Q-matrix and the skills mastered by the 5 examinees, the expected results are:

$$\mathbf{V} = \mathbf{WH} = \begin{matrix} & \text{examinees} \\ \text{items} & \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$

For example, taking the first item and the first examinee, we have, from \mathbf{W} , that item 1 requires skill 2, but, from \mathbf{H} , we see that examinee 1 only masters skill 1, therefore item 1 is failed by examinee 1. In fact, examinee 1's only success is over item 3 since all other items require either skills 2 or 3. It is important to emphasize that there are many solutions to $\mathbf{V} = \mathbf{WH}$. For example, the same results as those above

can be obtained with different Q-matrix and skills matrix:

$$\begin{matrix} & \text{examinees} \\ \text{items} & \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{matrix} = \begin{matrix} & \text{examinees} \\ \text{skills} & \begin{pmatrix} 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

Notice that the weights are changed as well as the ordering of rows and columns compared to the first solution. Nevertheless, it remains a valid factorization of \mathbf{V} that could be derived by some NMF algorithm.

Indeed, there are many NMF algorithms that were developed since its introduction by Lee and Seung [9] and they can yield different solutions. We refer the reader to [4] for a more thorough and recent review of this technique which has gained strong adoption in many different fields.

Whereas the other matrix factorization techniques often impose constraints of orthogonality among factors, NMF imposes the constraint that the two matrices, \mathbf{W} and \mathbf{H} , be non-negative. This constraint makes the interpretation much more intuitive in the context of using this technique for building a Q-matrix. It implies that the skills (latent factors) are additive “causes” that contribute to the success of items, and that they can only increase the probability of success and not decrease it, which makes good sense for skill factors. Note that negative values in \mathbf{W} can be interpreted as misconceptions and would lower the expected score to items, but allowing negative values in the factorization also opens up the space of possible solutions and raises the issue of convergence and of the multiplicity of solutions, making the interpretation of \mathbf{W} much more speculative.

The non-negative constraint and the additive property of the skills bring a specific interpretation of the Q-matrix. For example, if an item requires skills a and b with the same weight each, then each skill will contribute equally to the success of the item. This corresponds to the notion of a *compensatory* or *additive* model of skills.

In our study, we focus on high level skills, which we term *topic skills*. However, if an item requires two specific lower level skills, such as mastery of the rules $a/c + b/c = (a+b)/c$ and $a/b \cdot b = a$, a *conjunctive* model would be necessary, indicating that a failure is expected if any skill is not mastered. The standard interpretation of the Q-matrix corresponds to the conjunctive model, and the \mathbf{W} matrix of NMF does not correspond to this interpretation, unless and as mentioned, we assume that each item belongs to a single skill and for which case the two interpretations are indiscernible.

A last remark on NMF: as mentioned above, the factorization can produce multiple solutions, even with a single algorithm, which raises the issue of stability of the results. However, Schachtner et al. [11] discuss this issue and suggest that for binary data the problem may not appear at all. Nevertheless, we will assess the extent to which the multiple solution issue impacts the validity and usefulness of the approach by running multiple folds simulations.

3. Q-MATRIX EXTRACTION FROM REAL DATA

Winters et al. [13] experimented with NMF over SAT and Trivia tests data². We start by replicating their experiment and visualize the results graphically with a heatmap graph. This graph provides a global and intuitive view of the results. We turn to quantitative measures in the next section. The Trivia and SAT data sets have a similar structure: 40 question items broken down into 4 topics of 10 items each. Random samples of 100 examinees are taken from the complete data sets to run the Q-matrix extractions. This size is chosen as it represents realistic numbers of tests data that can be gathered in practice.

The 4 topics of the SAT data are (1) Mathematics, (2) Biology, (3) World History, and (4) French. These topics are sufficiently far apart that we can expect that the intra-topic correlation allows strong clustering. The data is composed of a total of 297 respondents who completed the 40 question items tests over the Internet. The profile of the respondents is unknown but they are probably from the university student community.

The trivia data set uses questions published from the Trivial Pursuit game. It mimics the structure of the SAT data set and is composed of 4 topics containing 10 questions on (1) Arts and entertainment, (2) Sports and leisure, (3) Science and nature, and (4) Geography.

The results of the NMF algorithm over these two data sets are reported in figure 1, where the heatmaps of the test results data and of the Q-matrix derived from the test results are displayed. Each of the four sub-figures is described below.

The SAT results are shown in figures 1(a) and 1(b). Figure 1(a) shows the raw data, where a dark red pixel represents a failure and a light yellow a success. Variation in the difficulty of each topic is apparent in this figure: items 1 to 10 show a higher success rate than items 10 to 20. Individual *item difficulty* is also apparent by the horizontal patterns. Vertical patterns are also apparent, especially within the items of a topic. This represents what we can call the examinee's *topic skill effect*. The pattern that spans across topics represents an examinee's *general ability effect*. We later refer to the *topic skill effect* as β_q , and to *item difficulty* and *general ability* as β_m and β_n respectively, in reference to the m rows and n columns of matrix \mathbf{V} in equation (1).

Figure 1(b) shows the Q-matrix extracted with NMF from the SAT data. As explained in section 2, the Q-matrix indicates the mapping of skills to items, and the higher values of the matrix tend towards the yellow colour whereas low values are red. A clustering consistent with the four SAT topics would show as four yellow clusters of 10 contiguous items, one for each skill. In reality, we see that clusters of the Mathematics (items 1 to 10) and the French items (31 to 40) are relatively well defined and apparent to the eye, but not as much with the Biology (11 to 20) and World History (21 to 30).

²The data sets from [13] were made available from alumni.cs.ucr.edu/~titus/. The simulation scripts of this article are available from www.professeurs.polymtl.ca/michel.desmarais/Papers/EDM2011/scripts.html. They are based on the NMF package from the statistical software R.

Figures 1(c) and 1(d) respectively show the raw data and the NMF derived Q-matrix for the Trivia data set. We see from the raw data graph that the success rate is lower than for the SAT data. As for the clustering, it is visually indiscernible in the Q-matrix. As we see later in table 1, clustering accuracy is in fact no better than chance.

These findings concur with that of Winters et al. [13] who found relatively good accuracy for the French and Mathematics topics of the SAT data, but found results that are no better than chance for the Trivia data. The most troubling findings from their experiments is that the Trivia results are similar to those obtained over a number of test outcome from different computer science courses: “Nearly every course behaves the same as the trivia data. Only our smallest data set, the Algorithms course data, showed any significant hint of topic structure.” This conclusion casts a gloomy picture for high level transfer models, where we aim to assess the mastery of topic specific skills from similar topic skills.

However, statistical characteristics of the data may also influence the clustering accuracy. For example, skewness of the scores towards 0% or 100% will result in sparsity of success/failure that can negatively affect the ability to extract a valid Q-matrix from the data. The Trivia data shows such skewness towards low success rate and we can question whether this is not the source of the low accuracy.

In the next section, we investigate the influence of the success rates and item and examinee variance over the Q-matrix validity using simulated data instead of real data.

4. Q-MATRIX EXTRACTION FROM SIMULATED DATA

Simulated data allows us to predefine the underlying item-skill mapping of the Q-matrix and generate data from this mapping. Using simulated data, we can ascertain under which assumptions the NMF can effectively derive back the Q-matrix.

4.1 Simulated data generation

To generate the simulated data, we create a probability matrix, \mathbf{P} , from which we generate a data matrix, \mathbf{D} , of the same dimensions as \mathbf{P} . The matrix \mathbf{D} corresponds to \mathbf{V} in equation (1) and it contains successes and failures, $\{0, 1\}$. The value of element d_{ij} in \mathbf{D} is randomly determined using p_{ij} from \mathbf{P} as the probability of success, and $1 - p_{ij}$ for failure.

For the sake maintaining the similarity with the real data analyzed in section 3, the dimensions of the probability and data matrices are 40×100 .

The probabilities of matrix \mathbf{P} are obtained according to the following model:

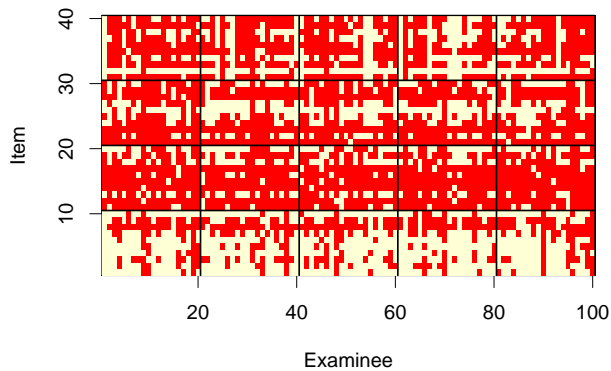
$$p_{ij} = \Phi(\beta_i + \gamma_j + \delta_{jk}) \quad (2)$$

Where:

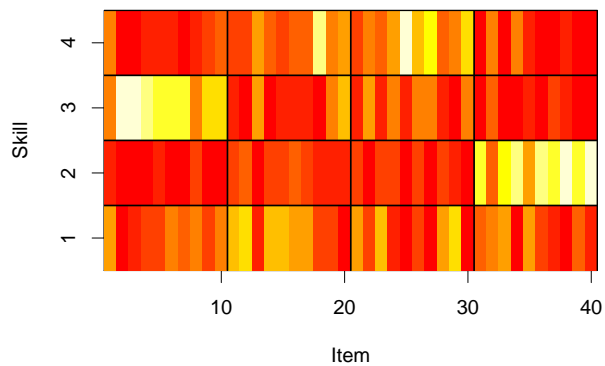
p_{ij} : represents the probability of success to item i by examinee j

$\Phi(x)$: is the *cumulative distribution function* of the standard normal distribution

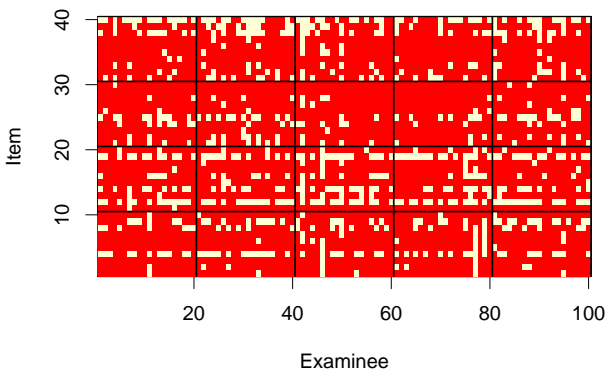
and where β_i , γ_j and δ_{jk} are random Gaussian variables representing respectively the item success rate (difficulty), examinee ability, and topic skill factors.



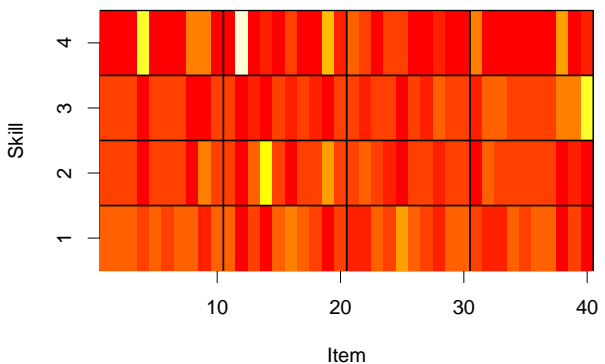
(a) Item outcome from SAT scores of 4 topics and a sample of 100 examinees.



(b) Image output of Q-matrix from NMF for 4 skills and 40 question items.



(c) Item outcome from Trivia scores of 4 topics and a sample of 100 examinees.



(d) Image output of Q-matrix from NMF for 4 skills and 40 question items.

Figure 1: NMF results over the SAT and Trivia data sets.

For every student j , his or her general ability γ_j is drawn from the distribution $\mathcal{N}(\bar{X}, s_n)$, where \bar{X} is the average score over all students and topics, and s_n the standard deviation of the average score per student. \bar{X} and s_n are taken from the real data, or fixed to given values depending on the experiment (see section 5). For every student j and skill category k , his or her topic specific ability δ_{jk} is drawn from the distribution $\mathcal{N}(\bar{X}_k, s_k)$, where \bar{X}_k is the mean score over category k , an s_k the standard deviation of the average score between categories. \bar{X}_k and s_k can again be fixed or taken from the real data. For every item i , the item difficulty β_i is sampled from the distribution $\mathcal{N}(\bar{X}, s_m)$ where \bar{X} is again the average score over all questions and students, and s_m the standard deviation of the per question average score.

The probability p_{ij} of student j answering question i from category k correctly is now computed according to equation (2). With probability p_{ij} , d_{ij} will be 1 (correct answer), otherwise it is 0 (incorrect answer).

Equation (2) can be considered as a simple model of examinee performance as a function of topic skill mastery, item

difficulty, and examinee general ability (which spans across topics). In spite of its simplicity compared to other means of generating simulated data (for an example, see [8]), it remains realistic for our context where we assume that topic skills are relatively independent, or at least this is an assumption we want to investigate and therefore it makes sense that our model follows that same assumption.

4.2 Clustering measure

Clustering is based on the simple algorithm which assigns each item to one of the 4 clusters based on the maximum column value in matrix \mathbf{W} . Given that we know the actual category of each item, the accuracy of the clustering can be computed. This is obtained by a two step process. First, a contingency table is compiled from the clustering algorithm. Next, the lines are reordered so that the sum of the diagonal is maximized. The ratio of this sum over the total represents the accuracy of the assignment. An example of the contingency table obtained is given below for the SAT data along

Table 1: Experiments over the parameter space of skills, items, and examinee (respectively δ_{jk} , β_i , and γ_j in equation (2)).

	Parameter space						N folds	Accuracy	
	Topic skill (δ_{jk})		Item (β_i)		Examinee (γ_j)			Mean acc.	S.d. acc.
	Mean	S.d.	Mean	S.d.	Mean	S.d.			
1*	0	0	0	1	0	1	20	0.36	0.05
2	0	0.10	0	1	0	1	20	0.48	0.07
3	0	0.25	0	1	0	1	20	0.60	0.11
4	0	0.50	0	1	0	1	20	0.93	0.08
5	0	1	0	1	0	1	20	1	0
6	0	1	0.50	1	0.50	1	20	1.00	0.01
7	0	1	1	1	1	1	20	0.98	0.07
8	0	1	1.50	1	1.50	1	20	0.90	0.12
9	0	1	2	1	2	1	20	0.81	0.16
<i>Trivia data parameters</i>									
10	0	0.12	-1.05	0.73	-1.05	0.45	20	0.75	0.12
11**	<i>n.a.</i>	<i>0.12</i>	<i>-1.05</i>	<i>0.73</i>	<i>-1.05</i>	<i>0.45</i>	<i>20</i>	<i>0.35</i>	<i>0.03</i>
<i>SAT data parameters</i>									
12	0	0.24	-0.33	0.86	-0.33	0.50	20	0.98	0.05
13**	<i>n.a.</i>	<i>0.24</i>	<i>-0.33</i>	<i>0.86</i>	<i>-0.33</i>	<i>0.50</i>	<i>20</i>	<i>0.72</i>	<i>0.02</i>
14***	<i>n.a.</i>	<i>0.24</i>	<i>-0.33</i>	<i>0.86</i>	<i>-0.33</i>	<i>0.50</i>	<i>20</i>	<i>0.96</i>	<i>0.05</i>

* No topic skill effect conditions
 ** Real data
 *** Real data and scoring for the Mathematics and French topics only

Comparing the results to the accuracy reported on experiments 11 and 13 for real data, we observe that for SAT data, the accuracy is lower than experiment 12 and somewhere between experiments 3 and 4, which corresponds to a standard deviation of topic skill between 0.25 and 0.5 when β_i and γ_j have a (0,1) standard distribution. In other words, the skill effect is a little less than half the item and examinee effects. If we look only at the clustering for Mathematics and French (experiment 14) which are the most separable topics, then the accuracy goes up to 0.96, which is much closer to experiment 12. In terms of relative effect, the skill effect between Mathematics and French is close to the 0.93 accuracy obtained in 4, for which the standard deviation of skill effect is 0.50 of the item and examinee parameters.

In summary, the Trivia data shows negligible effect of topic skill, whereas the SAT data shows an effect that is essentially attributable to the Mathematics and French topics that can be clearly distinguished in the Q-matrix derived with NMF. The topic skill effect can be quantified as somewhere between 1/4 to 1/2 of the item and examinee effect as measured by the standard deviation, and over 1/2 if we only take Mathematics and French effects alone.

6. DISCUSSION

We conjectured earlier that the low success rate of the Trivia data could explain the low accuracy results obtained. This is only partly the case. When the simulations parameters are set to the same values as the Trivia data, the accuracy obtained is 0.75 (experiment 10⁴) whereas the real data re-

⁴Experiment 10 has a relative skill-item s.e. of 0.12/0.73 = 0.16, standing between experiments 2 and 3, and a relative skill-examinee s.d. of 0.12/0.45 = 0.27, standing close to experiment 3. If the performance followed some additive function of each of these ratios, we would expect the performance to be no better than that of experiment 3, 0.60. Given that

sults are 0.35 (experiment 11). Therefore, results of experiment 10 suggest that the gap between 0.75 and 0.35 is attributable to the lack of skill effect in this data, or at least, the lack of the 4 Trivia topic skills effects. Alternatively, other skills may affect the results and they are by no means constrained to four as is the case for this study. Finally, we can also conjecture that the items belong to more than one skills and that the additive model of the Q-matrix investigated here is not appropriate (see for eg. [10] and [1]). These questions call for further investigation.

Our results further indicate that for well delineated topic skills like Mathematics and French, the effect is relatively strong, in a range around half that item difficulty and examinee ability according to the results in table 1, at least for highly separable topics like Mathematics and French. In this case, the accuracy of matching items to skills with NMF is well in the range of 90%, which confirms the effectiveness of this technique under these conditions.

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it stands higher at 0.75, we have to conclude that the effect of s.d. ratios over the performance is more complex, possibly a ratio of s.d. such as topic/(item × examinee).

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